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EFFECT OF SHEAR DEFORMATION ON OPTIMAL DESIGN OF ELASTIC BEAMS

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Abstract—In order to avoid unbounded local stresses in the optimal structure, a realistic optimal design must include the effect of shear deformation. In this paper, such effect on the optimal design of elastic beams is illustrated by a problem of a circular ring. The cross section of the ring is assumed to be rectangular with given depth and variable width. Timoshenko beam theory is employed in the analysis. It is found that the optimal ring with shear deformation is more uniform in its cross section distribution than the optimal ring without shear deformation. Thus, by including the effect of shear deformation, the efficiency of optimal design is reduced.

INTRODUCTION

A uniform method of treating a variety of problems of optimal design of structures was given by Prager and Taylor [1]. In their analysis, the structure is assumed to be of sandwich type, hence, the specific structural stiffness is a linear function of the specific structural weight.† It is found that in the minimum elastic compliance design, the specific strain energy per unit stiffness in the optimal structure remains constant throughout the whole structure. The same conclusion can also be drawn for the optimal design for maximum elastic buckling load where the specific strain energy is referred to the change in configuration due to buckling. Accordingly, in the problem of optimal design of sandwich beams, if we consider only the bending deformation of the beam and neglect the effect of transverse shear and axial force in the analysis, then the optimal beam would contain zero cross sections at locations of vanishing bending moment. For such designs, the local stresses would be infinite at zero cross sections when the axial force and transverse shear do not vanish.

In this paper, the effect of transverse shear deformation on optimal designs is illustrated by a problem of a circular ring deformed by two forces. For the purpose of simplification in analysis, the cross section of the ring is assumed to be rectangular with given depth and variable width. Thus, the cross-sectional area of the ring is considered as a design variable and the local bending stiffness and shear stiffness are proportional to the local cross-sectional area of the ring. Timoshenko beam theory [2] is employed in the analysis.

† The term "specific" is used as an abbreviation for "per unit length of a one-dimensional structure" or "per unit area of a two-dimensional structure".

BASIC EQUATIONS

Let us consider a circular curved beam with radius of the center line R. The fiber stress at any point in the beam with a distance z from the central surface is assumed to be

$$\bar{\sigma} = \frac{F}{A} + \frac{Mz}{I} \tag{1}$$

where F is the axial force, M is the bending moment, A is the cross-sectional area and I is the moment of inertia of the cross-sectional area about the neutral axis. The shear stress is

$$\bar{s} = \frac{S}{A} g(z) \tag{2}$$

where S is the total transverse shear and $g(z)$ is a function of z depending on the shape of the cross section. The strain (or stress) energy per unit length of the beam can be expressed by the following area integral over the cross section

$$U = \frac{1}{2}\int\left(\frac{\bar{\sigma}^2}{E} + \frac{\bar{s}^2}{G}\right) \mathrm{d}A = \frac{1}{2}\left(\frac{F^2}{AE} + \frac{M^2}{EI} + \frac{S^2}{kAG}\right) \tag{3}$$

where E and G are respectively Young's modulus and shear modulus of the beam and k is a constant defined by

$$\int g^2(z)\, \mathrm{d}A = \frac{A}{k}. \tag{4}$$

For rectangular cross sections, $k = \frac{2}{3}$.

Denote the distance measured along the center line of the beam by s. Let the inward normal distributed load be $p(s)$, the counterclockwise tangential distributed load $t(s)$, the inward normal displacement w and the counterclockwise tangential displacement v. By considering the equilibrium in moment, radial and tangential forces acting on an infinitesimal element of the beam as shown in Fig. 1, we have the following equations of equilibrium:

$$M' - S = 0, \tag{5}$$

$$S' + \frac{F}{R} + p = 0, \tag{6}$$

$$F' - \frac{M'}{R} + t = 0 \tag{7}$$

where prime represents the differentiation with respect to s.

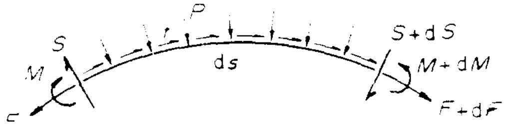

FIG. 1. Forces and moments on an infinitesimal arc of the beam.

The stress–displacement relations can be derived from the principle of minimum complementary energy with equations (5)–(7) as side conditions while the displacements w and v are considered to be prescribed. Thus, we obtain

$$\frac{M}{EI} = -\frac{v'}{R} - \varphi', \tag{8}$$

$$\frac{F}{AE} = -\frac{w}{R} + v', \tag{9}$$

$$\frac{S}{kAG} = w' - \varphi. \tag{10}$$

In equations (8) and (10), φ is a Lagrange multiplier function. Its geometrical meaning can be recognized by the special case of infinitesimal deformation of a straight beam for which $R = \infty$ and $F = v = 0$. In this case, equations (8) and (10) reduce to Timoshenko beam equations [2]:

$$\frac{M}{EI} = -\varphi', \tag{11}$$

$$\frac{S}{kAG} = w' - \varphi. \tag{12}$$

Therefore, φ is the angle of rotation of the cross section of the beam. Put $\psi = R\varphi$. Equations (8)–(10) can also be written as

$$\frac{M}{EI} = \frac{1}{R^2}(\ddot{v} + \dot{\psi}), \tag{13}$$

$$\frac{F}{AE} = -\frac{1}{R}(\dot{v} + w), \tag{14}$$

$$\frac{S}{kAG} = -\frac{1}{R}(\dot{w} + \psi) \tag{15}$$

where dot represents the differentiation with respect to the polar angle θ. Note that $(\dot{\ }) \equiv -R(\ )'$.

In the following, we shall consider the optimal design of beams for minimum overall compliance. The cross section of the beam is assumed to be rectangular with prescribed depth h and variable width. For this cross section, $I = \alpha^2 A$, where $\alpha^2 = h^2/12$ is a constant. The total volume of the beam is given as V. Hence,

$$\int A \, ds = V \tag{16}$$

The optimal design of beams of given height and variable width is equivalent to the design of sandwich beams. The necessary and sufficient condition of global optimum is given in [1]. It is

$$\frac{U}{A} = \text{const.} \tag{17}$$

OPTIMAL DESIGN OF A CIRCULAR RING FOR MINIMUM COMPLIANCE

A circular elastic slender ring of radius R and volume V is deformed by two equal and opposite forces P as shown in Fig. 2. The optimal ring is symmetrical with respect to $\theta = 0$ and $\theta = \pi/2$, hence, we need only consider one-quarter of the ring in the first quadrant $0 \leq \theta \leq \pi/2$.

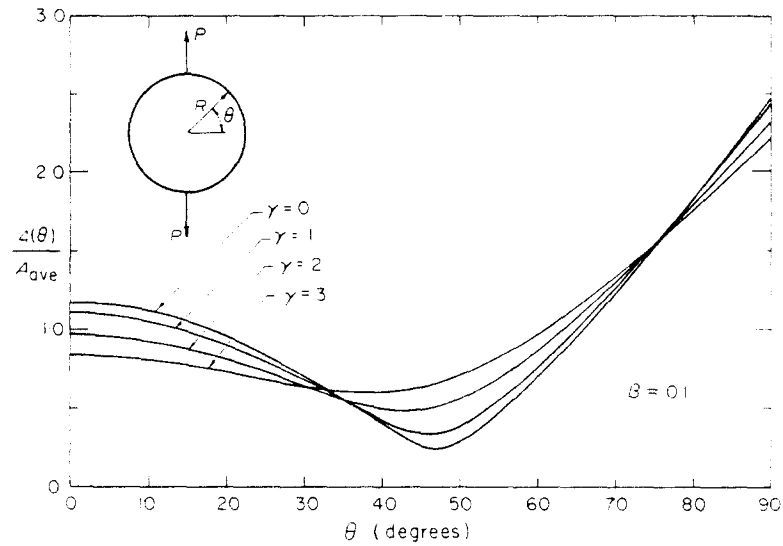

FIG. 2. $A(\theta)/A_{ave}$ curves for optimal circular rings.

Let the bending moment at $\theta = 0$ be M_1. The bending moment, axial force and transverse shear at any section in $0 \leq \theta \leq \pi/2$ are

$$M = -\frac{PR}{2}(\lambda - \cos\theta), \tag{18}$$

$$F = \frac{P}{2}\cos\theta, \tag{19}$$

$$S = \frac{P}{2}\sin\theta \tag{20}$$

where $\lambda = 1 - 2M_1/PR$ is an unknown constant to be determined.

The cross section of the optimal ring can be determined from the optimality condition, equation (17), using equations (3) and (18)–(20). It is

$$A = \frac{PR}{2Ec^{\frac{1}{2}}}\left[\frac{1}{\beta^2}(\lambda - \cos\theta)^2 + \cos^2\theta + \gamma^2\sin^2\right]^{\frac{1}{2}} \tag{21}$$

where $\beta^2 = \alpha^2/R^2$, $\gamma^2 = E/(kG)$ and c is a constant which can be determined from equation (16). Thus, equation (21) can be written as

$$A = \frac{V}{4Rf\beta}[(\lambda - \cos\theta)^2 + \beta^2\cos^2\theta + \beta^2\gamma^2\sin^2\theta]^{\frac{1}{2}} \tag{22}$$

where

$$f = \int_0^{\pi/2} \left[\frac{1}{\beta^2}(\lambda-\cos\theta)^2+\cos^2\theta+\gamma^2\sin^2\theta\right]^{\frac{1}{2}} d\theta. \tag{23}$$

From equations (13)–(15), (18)–(20) and (22), we have

$$\dot{v}+\dot{\psi} = -\frac{2PR^2f\beta}{EV}\frac{1}{\beta^2}[(\lambda-\cos\theta)^2+\beta^2\cos^2\theta+\beta^2\gamma^2\sin^2\theta]^{-\frac{1}{2}}(\lambda-\cos\theta), \tag{24}$$

$$\dot{v}+w = -\frac{2PR^2f\beta}{EV}[(\lambda-\cos\theta)^2+\beta^2\cos^2\theta+\beta^2\gamma^2\sin^2\theta]^{-\frac{1}{2}}\cos\theta, \tag{25}$$

$$\dot{w}+\psi = -\frac{2PR^2f\beta}{EV}\gamma^2[(\lambda-\cos\theta)^2+\beta^2\cos^2\theta+\beta^2\gamma^2\sin^2\theta]^{-\frac{1}{2}}\sin\theta. \tag{26}$$

The boundary conditions are

$$v(0) = \psi(0) = v(\pi/2) = \psi(\pi/2) = 0. \tag{27}$$

The solution of equations (24)–(27) leads to the condition for determination of λ

$$\int_0^{\pi/2} [(\lambda-\cos\theta)^2+\beta^2\cos^2\theta+\beta^2\gamma^2\sin^2\theta]^{-\frac{1}{2}}(\lambda-\cos\theta)\, d\theta = 0 \tag{28}$$

and the dimensionless compliance of the optimal ring

$$\delta_0 = -\frac{EVw(\pi/2)}{2PR^2} = f^2. \tag{29}$$

In Fig. 2, the ratio of $A(\theta)$ to its average value A_{ave} is plotted against θ for $\beta = 0{\cdot}1$ and various values of γ. When $\gamma = 0$, the effect of shear deformation is not included and

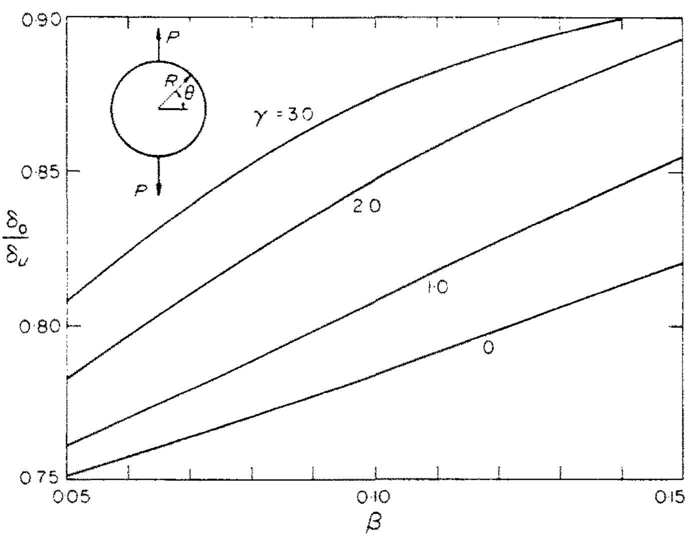

FIG. 3. Comparison of optimal rings with rings of uniform cross section.

the optimal design reduces to that given in [3]. It is found that the slopes of $A(\theta)/A_{ave}$ curves decrease with increasing γ.

In order to show the efficiency of the optimal design, the compliance δ_0 is compared with that of a ring with uniform cross section and identical volume. The compliance of the uniform ring can be obtained from Castigliano's theorem. It is

$$\delta_u = \pi\left[\frac{1}{\beta^2}\left(\frac{\pi}{8}-\frac{1}{\pi}\right)+(1+\gamma^2)\frac{\pi}{8}\right]. \tag{30}$$

In Fig. 3, the ratio δ_0/δ_u is plotted against β for different values of γ. When γ increases, the cross section distribution becomes more uniform and hence the value of δ_0/δ_u approaches the limiting value 1.

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Абстяакт—Чтобы избежать безграничные локальные напряжения в оптимальной конструкции, реяльный оптимальный расчет должен заключать еффект деформации сдвига. В настояшей работе илюстрируется этот эффкт при оптимальном расчете упругих балок на примере задечи круглого кольца. Предлагается, что поперечное сечение кольца прямоугольное, с заданной высотой и переменной шириной. В расчете используется теория балки Тимошенки. Определяется, что оптимальное кольцо с учетом деформации сдвига имеет более постоянное распределение напряжений в поперечным сечению, чем оптимальное кольцо, неучитаваюшее деформации сдвига. Затем учитывая эффект деформации сдвига, уменьшается трудоемкость оптимального расчета.